Some Results and Examples on Fredholm Alternative

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Part - 1

Method to find solution of Fredholm integral equation with examples

Fredholm Integral Equations

An equation of the form

$$\alpha(x)\, y(x) = f(x) + \lambda \int_a^b K(x,t)\, y(t)\, dt, \tag{1}$$

where α, f, K are given functions and λ, a, b are constants, is known as a **Fredholm integral equation**.

The function $y(x)$ is an unknown function to be determined.

When $f \equiv 0$, the equation (1) is called a **homogeneous Fredholm integral equation**.

Fredholm Integral Equations

The given function $K(x, t)$, which depends upon the variables x and t, is known as the **kernel** of the integral equation.

- when $\alpha \equiv 0$, equation (1) is known as a Fredholm integral equation of the **first kind**.
- when $\alpha \equiv 1$, the equation (1) is known as a Fredholm integral equation of the **second kind**.
- when α is a given function of x (not a constant function), then the equation (1) is known as a Fredholm integral equation of the **third kind**.

Fredholm Integral Equations

In general, when the function $\alpha(x)$ is positive throughout the interval (a, b), the equation (1) can be re-written in an equivalent form

$$\sqrt{\alpha(x)}\, y(x) = \frac{f(x)}{\sqrt{\alpha(x)}} + \lambda \int_a^b \frac{K(x,t)}{\sqrt{\alpha(x)\,\alpha(t)}} \sqrt{\alpha(t)}\, y(t)\, dt,$$

hence equation (1) can be considered an Fredholm integral equation of the second kind in the unknown function $\sqrt{\alpha(x)}\, y(x)$, with a modified kernel.

That is, if α has same sign in the integral (a, b), one can convert Fredholm integral equation of the third kind to second kind.

Separable or Degenerate Kernel (Simple Case)

A kernel $K(x, t)$ is called **separable** or **degenerate** if it can be expressed as the sum of a finite number of terms, each of which is the product of a function of x only and a function of t only. That is,

$$K(x, t) = \sum_{i=1}^{n} a_i(x) \, b_i(t),$$

where the functions $a_1(x), a_2(x), \ldots, a_n(x)$ and the functions $b_1(t), b_2(t), \ldots, b_n(t)$ are linearly independent.

Fredholm Integral Equations (with separable kernel)

With this kernel, the Fredholm integral equation of the second kind,

$$y(x) = f(x) + \lambda \int_a^b K(x,t)\; y(t)\; dt \tag{2}$$

becomes

$$y(x) = f(x) + \lambda \sum_{i=1}^{n} a_i(x) \int_a^b b_i(t)\, y(t)\; dt. \tag{3}$$

Substituting $c_i = \displaystyle\int_a^b b_i(t)\, y(t)\; dt$ in (2), we have a solution given by the formula

$$y(x) = f(x) + \lambda \sum_{i=1}^{n} c_i a_i(x), \tag{4}$$

and the problem is reduced to finding the c_i.

Substituting (4) in (3), we get

$$y(x) = f(x) + \lambda \sum_{i=1}^{n} a_i(x) \int_a^b b_i(t) \left\{ f(t) + \lambda \sum_{k=1}^{n} c_k a_k(t) \right\} dt.$$

Equating the above equation with the solution given by the formula (4), we get

$$f(x) + \lambda \sum_{i=1}^{n} c_i a_i(x) = f(x) + \lambda \sum_{i=1}^{n} a_i(x) \int_a^b b_i(t) \left\{ f(t) + \lambda \sum_{k=1}^{n} c_k a_k(t) \right\} dt.$$

$$\implies \sum_{i=1}^{n} a_i(x) \left\{ c_i - \int_a^b b_i(t) \left\{ f(t) + \lambda \sum_{k=1}^{n} c_k a_k(t) \right\} dt \right\} = 0.$$

Fredholm Integral Equations

Since functions $a_i(x)$ are linearly independent; therefore

$$c_i - \int_a^b b_i(t) \left\{ f(t) + \lambda \sum_{k=1}^{n} c_k a_k(t) \right\} dt = 0, \quad i = 1, \ldots, n. \tag{5}$$

Denoting

$$\int_a^b b_i(t) f(t)\, dt = f_i, \quad \int_a^b b_i(t) a_k(t)\, dt = a_{ik}, \tag{6}$$

where f_i and a_{ik} are known constants, equation (5) becomes

$$c_i - f_i - \lambda \sum_{k=1}^{n} a_{ik} c_k = 0, \quad i = 1, \ldots, n$$

and hence

$$c_i - \lambda \sum_{k=1}^{n} a_{ik} c_k = f_i, \quad i = 1, \ldots, n. \tag{7}$$

Fredholm Integral Equations

For $i = 1, 2, \dots, n$, we have a system of n algebraic equations for the unknowns c_i.

$$\begin{aligned}
c_1 - \lambda c_1 a_{11} - \lambda c_2 a_{12} - \cdots - \lambda c_n a_{1n} &= f_1 \\
c_2 - \lambda c_1 a_{21} - \lambda c_2 a_{22} - \cdots - \lambda c_n a_{2n} &= f_2 \\
\vdots \quad - \quad \vdots \quad - \quad \vdots \quad - \quad \vdots \quad - \quad \vdots \quad &= \vdots \\
c_n - \lambda c_1 a_{n1} - \lambda c_2 a_{n2} - \cdots - \lambda c_n a_{nn} &= f_n
\end{aligned}$$

$$\Longrightarrow \begin{pmatrix} 1-\lambda a_{11} & -\lambda a_{12} & \dots & -\lambda a_{1n} \\ -\lambda a_{21} & 1-\lambda a_{22} & \dots & -\lambda a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ -\lambda a_{n1} & 1-\lambda a_{n2} & \dots & 1-\lambda a_{nn} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} \qquad (8)$$

$$\Longrightarrow (1 - \lambda A)C = F.$$

Case : $f \equiv 0$

Recall that $y(x) = f(x) + \lambda \sum_{i=1}^{n} c_i a_i(x)$.

If the function $f(x)$ is identically zero, (it is the **homogeneous Fredholm integral equation**), so each $f_i = 0$ and hence, $F = 0$. Moreover,

1. $c_1 = c_2 = \cdots = c_n = 0$ when $\det(I - \lambda A) \neq 0$. Hence the equation possesses the trivial solution $y \equiv 0$ (**unique solution**).
2. However, if $\det(I - \lambda A) = 0$, at least one of the c_i's can be assigned arbitrarily, and the remaining c_j' s can be accordingly determined. In this cases, **infinitely many solutions** of the integral equation exist. Inverses of those values of λ for which $\det(I - \lambda A) = 0$ are known as **eigenvalues** and any nontrivial solution of the homogeneous integral equation is called a **corresponding eigenfunction**.

Case : $f \not\equiv 0$ but f is orthogonal to each b_i, $i = 1, 2, \ldots, n$

Recall that $y(x) = f(x) + \lambda \sum_{i=1}^{n} c_i a_i(x)$.

If the function $f(x)$ is not identically zero and f is orthogonal to each b_i, $i = 1, 2, \ldots, n$, so each $f_i = 0$ and hence, $F = 0$. Moreover,

1. $c_1 = c_2 = \cdots = c_n = 0$ when $\det(I - \lambda A) \neq 0$. Hence the equation possesses the **unique solution** $y(x) = f(x)$.
2. However, if $\det(I - \lambda A) = 0$, at least one of the c_i's can be assigned arbitrarily, and the remaining c_j' s can be accordingly determined. In this cases, **infinitely many solutions** of the integral equation exist.

Case : $f \neq 0$ and some b_i is not orthogonal to f

If the function $f(x)$ is not identically zero and some b_i $(i = 1, 2, \ldots n)$ is not orthogonal to f, then

1. The equation possesses **unique solution**, $C = (I - \lambda A)^{-1} F$ when $\det(I - \lambda A) \neq 0$.
2. Suppose $\det(I - \lambda A) = 0$. There are two cases :
   (a) there is **no solution** if $rank(I - \lambda A)$ and $rank\{(I - \lambda A \mid F)\}$ are different.
   (b) there are **infinitely many solutions** if $rank(I - \lambda A)$ and $rank\{(I - \lambda A \mid F)\}$ are the same.

Example 1.

Solve the Fredholm integral equation of the second kind

$$y(x) = x + \lambda \int_0^1 (xt^2 + x^2 t)\ y(t)\, dt. \tag{9}$$

Solution: The kernel $k(x,t) = xt^2 + x^2 t$ is separable and we can set

$$c_1 = \int_0^1 t^2\ y(t) dt, \quad c_2 = \int_0^1 t\ y(t) dt,$$

Then (9) becomes

$$y(x) = x + \lambda c_1 x + \lambda c_2 x^2.$$

On putting this value in c_1 and c_2, we obtain

$$c_1 = \frac{1}{4} + \frac{1}{4}\lambda c_1 + \frac{1}{5}\lambda c_2,$$
$$c_2 = \frac{1}{3} + \frac{1}{3}\lambda c_1 + \frac{1}{4}\lambda c_2.$$

Now, after finding the values of c_1 and c_2, we get the solution

$$\begin{aligned} y(x) &= x + \lambda c_1 x + \lambda c_2 x^2 \\ &= \frac{240x - 60\lambda x + 80\lambda x^2}{240 - 120\lambda - \lambda^2}. \end{aligned}$$

Example 2.

Solve the integral equation

$$y(x) = \lambda \int_0^1 (3x - 2)t \; y(t)dt. \tag{10}$$

Solution: Note that the given equation is a homogeneous Fredholm integral equation.

Let

$$c = \int_0^1 t \; y(t)dt.$$

Then (10) is reduced to

$$y(x) = \lambda c(3x - 2).$$

Fredholm Integral Equations

We obtain

$$c = \int_0^1 \lambda ct(3t-2)dt = \lambda \int_0^1 (3t^2 - 2t)\ dt = 0,$$

hence $y(x) = 0$, which is a zero solution. Therefore, the given integral equation does not possess any eigenvalue or eigenfunction.

Note that here A is the zero matrix and $det(I - \lambda A) = 1 \neq 0$.

Example 3.

Consider the differential equation

$$y(x) = f(x) + \lambda \int_0^1 (1 - 3xt)\ y(t)dt.$$

This equation can be written in the form

$$y(x) = f(x) + \lambda(c_1 - 3c_2 x)$$

where $c_1 = \int_0^1 \ y(t)dt$ and $c_2 = \int_0^1 t\ y(t)dt.$

On solving, we get

$$c_1 = \lambda(c_1 - \frac{3}{2}c_2) + \int_0^1 f(t)\,dt,$$
$$c_2 = \lambda(\frac{1}{2}c_1 - c_2) + \int_0^1 tf(t)dt,$$

or

$$(1-\lambda)c_1 + \frac{3}{2}\lambda c_2 = \int_0^1 f(t)dt,$$
$$-\frac{1}{2}\lambda c_1 + (1+\lambda)c_2 = \int_0^1 tf(t)dt.$$

The determinant of $(I - \lambda A)$ is given by

$$D(\lambda) = \frac{4 - \lambda^2}{4}.$$

It follows that a unique solution exists if and only if

$$\lambda \neq \pm 2.$$

Fredholm Integral Equations

Suppose $f \equiv 0$. There are two cases:

1. If $\lambda \neq \pm 2$ (determinant is non-zero), the only solution is the trivial solution $y(x) = 0$.
2. If $\lambda = \pm 2$, we have a non-zero solution. Then $\pm 1/2$ are the eigen values of A.

Fredholm Integral Equations

If $\lambda = +2$, the system is reduced to

$$-c_1 + 3c_2 = \int_0^1 f(t)\,dt,$$
$$-c_1 + 3c_2 = \int_0^1 tf(t)\,dt.$$

The system is compatible only if the function $f(x)$ satisfies the condition

$$\int_0^1 f(t)dt = \int_0^1 tf(t)dt \quad \text{or} \quad \int_0^1 (1-t)f(t)dt = 0.$$

If the above condition is satisfied, the corresponding system is consistent, hence the integral has a solution.

If $\lambda = -2$, the system is reduced to

$$c_1 - c_2 = \frac{1}{3}\int_0^1 f(t)\,dt,$$
$$c_1 - c_2 = \int_0^1 tf(t)dt.$$

The system is compatible only if the function $f(x)$ satisfies the condition

$$\frac{1}{3}\int_0^1 f(t)dt = \int_0^1 tf(t)dt \quad \text{or} \quad \int_0^1 (1-3t)f(t)dt = 0.$$

If the above condition is satisfied, the corresponding system is consistent, hence the integral has a solution.

First let us consider the case when $f(x) = 0$.

If $\lambda \neq \pm 2$, the only solution is the trivial solution.

If $\lambda = 2$, the system gives $c_1 = 3c_2$. Thus the solution is

$$y(x) = 2c_1(1 - x) = c(1 - x)$$

where c is an arbitrary constant. The function $(1 - x)$ and all its non-zero multiples are the eigen function corresponding to the eigen value $\lambda = 1/2$.

If $\lambda = -2$, the system gives $c_1 = c_2$. Thus the solution is

$$y(x) = 2c_1(1 - 3x) = d(1 - 3x)$$

where d is an arbitrary constant.

The function $(1 - 3x)$ and all its non-zero multiples are the eigen function corresponding to the eigen value $\lambda = -1/2$.

In the non-homogeneous case, $f(x) \neq 0$, a unique solution exists if $\lambda \neq \pm 2$.

If $\lambda = 2$, the algebraic system shows that no solution exists unless $f(x)$ is orthogonal to $1 - x$ over the interval $(0, 1)$, i.e., unless $f(x)$ is orthogonal to the eigen function corresponding to $\lambda = 2$.

If f satisfies the orthogonality condition, then both linear equations are equivalent. Hence we obtain

$$c_1 = 3c_2 - \int_0^1 f(t)\, dt,$$

Fredholm Integral Equations

That gives the solution as follows:

$$\lambda = 2 : \quad y(x) = f(x) - 2\int_0^1 f(t)\,dt + c(1-x)$$

when

$$\int_0^1 (1-x) f(x)\,dx = 0. \tag{11}$$

where c is an arbitrary constant. Thus in this case, infinitely many solutions exist, differing by a multiple of relevant eigen function.

Fredholm Integral Equations

Similarly, if $\lambda = -2$ there is no solution unless $f(x)$ is orthogonal to $(1-3x)$ over $(0,1)$ in which case infinitely many solutions exist as follows:

$$\lambda = -2: \quad y(x) = f(x) - \frac{2}{3}\int_0^1 f(t)\,dt + d\,(1-3x),$$

where

$$\int_0^1 (1-3x)\,f(t)\,dt = 0 \tag{12}$$

where d is an arbitrary constant.

Example 4.

Discuss solution of the integral equation

$$y(x) = f(x) + \lambda \int_0^{2\pi} \sin(x+t)\ y(t)dt$$

and show that the integral equation

$$y(x) = f(x) + \frac{1}{\pi} \int_0^{2\pi} \sin(x+t)\ y(t)dt$$

has no solution when $f(x) = x$, *and*

has infinitely many solutions when $f \equiv 1$.

Fredholm Integral Equations

Here $K(x,t) = \sin(x+t) = \sin x \cos t + \cos x \sin t$.

The corresponding matrix equation $(I - \lambda A)C = F$ becomes

$$\begin{pmatrix} 1 & -\lambda\pi \\ -\lambda\pi & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \int_0^{2\pi} \cos t \ f(t) \ dt \\ \int_0^{2\pi} \sin t \ f(t) \ dt \end{pmatrix}.$$

Also, $det(I - \lambda A) = 1 - \lambda^2\pi^2$.

When $det(I - \lambda A) \neq 0$, the integral equation has a unique solution.

When $det(I - \lambda A) = 0$, that is, $\lambda = \pm 1/\pi$, the given integral equation will either have no solution or have infinitely many solutions.

Fredholm Integral Equations

Now we first solution to the homogeneous integral equation

$$y(x) = \lambda \int_0^{2\pi} \sin(x+t)\; y(t)dt.$$

The corresponding algebraic system is

$$\begin{aligned} c_1 - \lambda\pi c_2 &= 0 \\ -\lambda\pi c_2 + c_2 &= 0. \end{aligned}$$

When $\lambda = 1/\pi$, we obtain $c_1 = c_2$, and hence

$$y(x) = c(\sin x + \cos x), \quad \text{where } c \text{ is an arbitrary constant.}$$

When $\lambda = -1/\pi$, we obtain $c_1 = -c_2$, and hence

$$y(x) = d(\sin x - \cos x), \quad \text{where } d \text{ is an arbitrary constant.}$$

Fredholm Integral Equations

Recall that $\begin{pmatrix} 1 & -\lambda\pi \\ -\lambda\pi & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \int_0^{2\pi} \cos t \; f(t) \; dt \\ \int_0^{2\pi} \sin t \; f(t) \; dt \end{pmatrix}.$

When $\lambda = 1/\pi$, necessary condition for the system

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \int_0^{2\pi} \cos t \; f(t) \; dt \\ \int_0^{2\pi} \sin t \; f(t) \; dt \end{pmatrix}.$$

to be consistent is that

$$\int_0^{2\pi} f(t) \; (\sin t + \cos t) \; dt = 0.$$

Fredholm Integral Equations

When $\lambda = -1/\pi$, necessary condition for the system

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \int_0^{2\pi} \cos t \; f(t) \; dt \\ \int_0^{2\pi} \sin t \; f(t) \; dt \end{pmatrix}.$$

to be consistent is that

$$\int_0^{2\pi} f(t) \; (\sin t - \cos t) \; dt = 0.$$

Observations

The given integral equation

$$y(x) = f(x) + \frac{1}{\pi} \int_0^{2\pi} \sin(x+t)\; y(t) dt$$

1. has no solution when $f(x) = x$ because

$$\int_0^{2\pi} f(t)\; (\sin t - \cos t)\; dt \neq 0.$$

2. has infinitely many solutions when $f \equiv 1$ because

$$\int_0^{2\pi} f(t)\; (\sin t - \cos t)\; dt = 0.$$

Observations

Thus the integral equation will possess infinitely many solutions given by

$$y(x) = 1 + c(\sin x + \cos x) + d(\sin x - \cos x).$$

That is,

$$y(x) = 1 + A\cos x + B\sin x,$$

where A and B are arbitrary constants.

Part - 2

Solution of the Integral Equation Using Functional Analysis Techniques

Solution of the Integral Equation Using Functional Analysis Techniques

Let $v(t)$ and $w(t)$ be continuous functions on $[a, b]$.

Consider an integral equation of the form

$$x(t) = y(t) + v(t) \int_a^b w(s)\, x(s)\, ds. \tag{13}$$

This integral equation comes up frequently in applications.

We shall first discuss a method to solve the integral equation which leads to a result.

Overview

The discussion of solving the integral equation is useful to generalize the result, even for compact operators on normed spaces.

The generalized result is given as follows and is called "**Fredholm Alternative**". At the end of the lecture, we shall prove the generalized result.

Theorem 5 (Fredholm Alternative).

Let X be a Banach space and let K be an operator in $K(X)$. Set $A = I - K$. Then, $R(A)$ is closed in X and $\dim N(A) = \dim N(A^*)$ is finite. In particular, either $R(A) = X$ and $N(A) = \{0\}$, or $R(A) \neq X$ and $N(A) \neq \{0\}$.

Solution of the Integral Equation

Let $v(t)$ and $w(t)$ be continuous functions on $[a, b]$.

Consider an integral equation of the form

$$x(t) = y(t) + v(t) \int_a^b w(s)\, x(s)\, ds. \tag{14}$$

Let $X = C[a, b]$, with sup-norm.

For a given continuous function $y(t)$ on $[a, b]$, the problem is to find a solution $x(t)$ in X.

Solution of the Integral Equation

Define $x_w^* : X \to \mathbb{K}$ by

$$x_w^*(x) = \int_a^b w(s)\ x(s)\ ds. \tag{15}$$

As $|x_w^*(x)| \leq c\|x\|_\infty$, where $c = \int_a^b |w(s)|\ ds$, hence $x_w^* \in X^*$.

We are now having an element v in X and x_w^* in X^* and $K : X \to X$ is an operator on X defined by

$$(Kx)(t) = x_w^*(x)\ v(t) \tag{16}$$

for the operator equation

$$x = y + Kx. \tag{17}$$

Exercise 6.

Show that K is a linear bounded, rank-one operator.

Solution of the Integral Equation

Now clearly, in order to solve

$$x = y + Kx,$$

it suffices to find Kx, that is, to find the scalar $x_w^*(x)$.

Since $x = y + Kx$, $x_w^*(x) = x_w^*(y) + x_w^*(Kx)$ implies

$$x_w^*(x)[1 - x_w^*(v)] = x_w^*(y). \tag{18}$$

Case 1 : when $x_w^*(v) \neq 1$

When $x_w^*(v) \neq 1$,

$$x_w^*(x) = \frac{x_w^*(y)}{1 - x_w^*(v)} \quad \text{hence} \quad Kx = \frac{x_w^*(y)}{1 - x_w^*(v)} v.$$

Thus if $x_w^*(v) \neq 1$, we have a solution

$$x(t) = y(t) + \frac{x_w^*(y)}{1 - x_w^*(v)} v(t).$$

Case 1 : when $x_w^*(v) \neq 1$

Concerning **uniqueness**, we see from that $\boxed{x_w^*(x)[1 - x_w^*(v)] = x_w^*(y)}$ if $y = 0$, then $x_w^*(x) = 0$, and hence, so $x = 0$.

Hence the **unique solution** of the given integral equation is

$$x(t) = y(t) + \frac{\int_a^b w(s)\ y(s)\ ds}{1 - \int_a^b w(s)\ v(s)\ ds} v(t)$$

provided $\int_a^b w(s)\ v(s)\ ds \neq 1$.

Note that there is no condition on y when there is a unique solution. But the condition is that the image of v under x_w^* is not equal to 1.

Case 2 : when $x_w^*(v) = 1$

Suppose $x_w^*(v) = 1$.

By the equation $\boxed{x_w^*(x)[1 - x_w^*(v)] = x_w^*(y)}$ we get that $x_w^*(y) = 0$, in order that the given integral equation has a solution.

So let us assume that

$$x_w^*(y) = \int_a^b w(s)\ y(s)\ ds = 0$$

then $x_w^*(x)$ can be any scalar, so that the equation

$$x = y + Kx = y + x_w^*(x)v$$

has many solutions provided $x_w^*(y) = 0$.

Rank-One Operator

We discussed solutions of the integral equation of the form

$$x(t) = y(t) + v(t) \int_a^b w(s)\, x(s)\, ds \tag{19}$$

where $y(t)$ and $v(t)$ are given continuous functions on $[a, b]$.

The discussion leads to the following result.

Theorem 7.

Let X be a normed space and let $A = I - K$, where K is of the form

$$Kx = x_1^*(x)x_1$$

where x_1 is a given element of X and x_1^* is a given element of X^*.

If $N(A) = \{0\}$, then $R(A) = X$. Otherwise, $R(A)$ is closed in X, and $N(A)$ if finite dimensional having the same dimension as $N(A^*)$. FA-1(P-1)T-1

Outline of the proof

If x_1^* or x_1 is zero, the proof is obvious. Hence we assume that both are non-zero.

In order to solve

$$Ax = x - Kx = y,$$

it suffices to find Kx, that is, to find the scalar $x_1^*(x)$.

Since $x = y + Kx$, $x_1^*(x) = x_1^*(y) + x_1^*(Kx)$ implies

$$x_1^*(x)[1 - x_1^*(x_1)] = x_1^*(y).$$

Case 1 : When $x_1^*(x_1) \neq 1$, what is $N(A)$?

Suppose $x \in N(A)$. Then $x = Kx$, so

$$x = \alpha x_1 \quad \text{for some} \quad \alpha.$$

Now, we have

$$\alpha x_1 = x = Kx = K(\alpha x_1) = \alpha x_1^*(x_1) x_1$$

implies

$$\alpha \Big[1 - x_1^*(x_1) \Big] x_1 = 0.$$

Since $x_1^*(x_1) \neq 1$, α must be zero. Thus $N(A) = \{0\}$ so A is one-to-one.

Case 1 : When $x_1^*(x_1) \neq 1$, what is $R(A)$?

When $x_1^*(x_1) \neq 1$,

$$x_1^*(x) = \frac{x_x^*(y)}{1 - x_1^*(x_1)} \quad \text{hence} \quad Kx = \frac{x_1^*(y)}{1 - x_1^*(x_1)} x_1.$$

Hence if $x_1^*(x_1) \neq 1$, we have a solution

$$x = y + \frac{x_1^*(y)}{1 - x_1^*(x_1)} x_1.$$

For any $y \in X$, if $x_1^*(x_1) \neq 1$, then there is a unique solution x for the operator equation

$$Ax = y.$$

Thus $R(A) = X$ so A is onto.

Case 1 : When $x_1^*(x_1) \neq 1$, what is $N(A^*)$?

We use I to denote the identity operator on X^* as well. By the definition of adjoint of K,

$$\begin{aligned}(K^*x^*)(x) &= x^*(Kx)\\ &= x_w^*(x)x_w^*(v).\end{aligned}$$

Suppose $x^* \in N(A^*)$. Then $x^* = K^*x^*$, so

$$x^* = \beta x_1^* \quad \text{for some} \quad \beta.$$

Now, we have

$$\beta x_1^* = x^* = K^*x^* = K^*(\beta x_1^*) = \beta x^*(x_1)x_1^*$$

implies

$$\beta\Big[1 - x_1^*(x_1)\Big]x_1^* = 0.$$

Since $x_1^*(x_1) \neq 1$, β must be zero. Thus $N(A^*) = \{0\}$ so A^* is one-to-one.

Case 2 : When $x_1^*(x_1) = 1$, what is $N(A)$?

Suppose $x \in N(A)$. Then $x = Kx$, so

$$x = \alpha x_1 \quad \text{for some} \quad \alpha.$$

Now, we have

$$\alpha x_1 = x = Kx = K(\alpha x_1) = \alpha x_1^*(x_1) x_1$$

implies

$$\alpha \Big[1 - x_1^*(x_1) \Big] x_1 = 0.$$

Since $x_1^*(x_1) = 1$, α can be any scalar. Thus $N(A) = span\{x_1\}$.

Case 2 : When $x_1^*(x_1) = 1$, what is $R(A)$?

Let $y \in X$.

In order to solve

$$Ax = x - Kx = y,$$

it suffices to find Kx, that is, to find the scalar $x_1^*(x)$.

Since $x = y + Kx$, $x_1^*(x) = x_1^*(y) + x_1^*(Kx)$ implies

$$x_1^*(x)[1 - x_1^*(x_1)] = x_1^*(y).$$

If $x_1^*(x_1) = 1$, then $x_1^*(y)$ has to be zero.

To have a solution for $Ax = y$, the element y cannot be an arbitrary element in X, but it has to satisfy that $x_1^*(y) = 0$. In this case, $x_1^*(x)$ is chosen to be any scalar, hence there are several solutions for y.

Case 2 : When $x_1^*(x_1) = 1$, what is $R(A)$?

In other words, we can solve $Ax = y$ only for those y in the set

$$\{y : x_1^*(y) = 0\} = {}^{\perp}\{x_1^*\} \quad \left[\text{the annihilator of } \{x_1^*\}\right].$$

Hence ${}^{\perp}\{x_1^*\} \subseteq R(A)$.

On the other hand, let $y \in R(A)$, then $y = Ax$ for some $x \in X$. As $x_1^*(x_1) = 1$ and $Ax = y$ has a solution, then $y \in^{\perp} \{x_1^*\}$.

Thus

$$R(A) = {}^{\perp}\{x_1^*\}.$$

Case 2 : When $x_1^*(x_1) = 1$, what is $N(A^*)$?

We use I to denote the identity operator on X^* as well. By the definition of adjoint of K,

$$(K^*x^*)(x) = x^*(Kx) = x_w^*(x)x_w^*(v).$$

Suppose $x^* \in N(A^*)$. Then $x^* = K^*x^*$, so $x^* = \beta x_1^*$, for some β.

Now, we have

$$\beta x_1^* = x^* = K^*x^* = K^*(\beta x_1^*) = \beta x^*(x_1)x_1^*$$

implies

$$\beta\Big[1 - x_1^*(x_1)\Big]x_1^* = 0.$$

Since $x_1^*(x_1) = 1$, β can be any scalar. Thus $N(A^*) = span\{x_1^*\}$.

Finite Rank Operator

Next we consider an operator of finite rank. Let the operator K be of the form

$$Kx = \sum_{j=1}^{n} x_j^*(x)x_j, \quad x_j \in X, x_j^* \in X^*.$$

Theorem 8.

Let X be a normed space, and let K be an operator of finite rank on X. Set $A = I - K$. Then $R(A)$ is closed in X, and the dimensions of $N(A)$ and $N(A^*)$ are finite and equal. FA-1(P-3)T-2

Proof

Let $\{x_1, x_2, \ldots, x_n\}$ be a basis for $R(K)$. For $x \in R(K)$, we have

$$x = \sum_{j=1}^{n} \alpha_j x_j$$

for some scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ (depending on Kx). Let's write

$$x = \sum_{j=1}^{n} \alpha_j(Kx) x_j.$$

We first claim that any bounded finite rank operator $K : X \to X$ is of the form

$$Kx = \sum_{j=1}^{n} x_j^*(x) x_j, \quad \text{for some} \;\; x_j \in X, x_j^* \in X^*.$$

Proof (contd...)

Let $x \in X$. Since $\{x_1, x_2, \ldots, x_n\}$ is a basis for $R(K)$, so

$$x = \sum_{j=1}^{n} \alpha_j(Kx)x_j.$$

Since $R(K)$ is finite dimensional, the norms on $R(K)$ are equivalent.

In particular, $\|Kx\| := \sum_{j=1}^{n} |\alpha_i(Kx)|$ and $\|Kx\|$ are equivalent.

Hence there exists a constant $C > 0$ such that

$$\sum_{j=1}^{n} |\alpha_j(Kx)| \leq C\|Kx\|.$$

Proof (contd...)

Since K is bounded,

$$\sum_{j=1}^{n} |\alpha_j(Kx)| \leq C\|Kx\| \leq C\|K\|.\|x\|,$$

so α_j is a bounded linear functional on $R(K)$.

By Hahn-Banach Theorem, there are functionals $x_j^* \in X^*$ such that

$$\alpha_j(x) = x_j^*(x), \quad \text{for all} \;\; x \in X.$$

Hence $K : X \to X$ is of the form

$$Kx = \sum_{j=1}^{n} x_j^*(x)x_j, \quad \text{for some} \;\; x_j \in X, x_j^* \in X^*.$$

We may take x_j and x_j^* are linearly independent in the expression. When they are not linearly independent, combine them.

Proof (contd...)

By the definition of adjoint of K, K^* is of the form

$$K^*x^* = \sum_{k=1}^{n} x^*(x_k)x_k^*.$$

Case 1 : What is $N(A)$?

Suppose $x \in N(A)$. Then $x = Kx$, so $x = \sum_{j=1}^{n} \alpha_j x_j$, for some scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$.

$$\sum_{j=1}^{n} \alpha_j x_j = x = Kx = \sum_{j=1}^{n} x_j^*(x) x_j = \sum_{j=1}^{n} x_j^* \Big(\sum_{k=1}^{n} \alpha_k x_k \Big) x_j$$

which implies that $\displaystyle\sum_{j=1}^{n} \Big\{ \alpha_j - \sum_{k=1}^{n} \alpha_k x_j^*(x_k) \Big\} x_j = 0$. Since $\{x_1, x_2, \ldots, x_n\}$ is linearly independent, for each $j = 1, 2, \ldots, n$, we have $\alpha_j - \sum_{k=1}^{n} \alpha_k x_j^*(x_k) = 0$. Hence

$$\begin{pmatrix} 1 - x_1^*(x_1) & -x_1^*(x_2) & \cdots & -x_1^*(x_n) \\ -x_2^*(x_1) & 1 - x_2^*(x_2) & \cdots & -x_2^*(x_n) \\ \vdots & \vdots & \cdots & \vdots \\ -x_n^*(x_1) & -x_n^*(x_2) & \cdots & 1 - x_n^*(x_n) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Case 1 : When $\det \Delta \neq 0$, what is $N(A)$?

Let

$$\Delta = \begin{pmatrix} 1 - x_1^*(x_1) & -x_1^*(x_2) & \cdots & -x_1^*(x_n) \\ -x_2^*(x_1) & 1 - x_2^*(x_2) & \cdots & -x_2^*(x_n) \\ \vdots & \vdots & \cdots & \vdots \\ -x_n^*(x_1) & -x_n^*(x_2) & \cdots & 1 - x_n^*(x_n) \end{pmatrix}.$$

We have,

$$\Delta \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since $\det \Delta \neq 0$, we must have all α_j's are zero. Thus $N(A) = \{0\}$ so A is one-to-one.

Case 1 : When $\det \Delta \neq 0$, what is $N(A^*)$?

Suppose $x^* \in N(A^*)$. Then $x^* = K^* x^*$, so $x^* = \sum_{j=1}^{n} \beta_j x_j$, for some scalars $\beta_1, \beta_2, \ldots, \beta_n$.

$$\sum_{j=1}^{n} \beta_j x_j^* = x^* = K^* x^* = \sum_{j=1}^{n} x^*(x_j) x_j^* = \sum_{j=1}^{n} \Big(\sum_{k=1}^{n} \beta_k x_k^* \Big)(x_j) x_j^*$$

which implies that

$$\sum_{j=1}^{n} \Big\{ \beta_j - \sum_{k=1}^{n} \beta_k x_k^*(x_j) \Big\} x_j^* = 0.$$

Since $\{x_1^*, x_2^*, \ldots, x_n^*\}$ is linearly independent, for each $j = 1, 2, \ldots, n$,

$$\beta_j - \sum_{k=1}^{n} \beta_k x_k^*(x_j) = 0.$$

Case 1 : When $\det \Delta \neq 0$, what is $N(A^*)$?

Hence we have,

$$\Delta^T \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since $\det \Delta \neq 0$, we must have all β_j's are zero. Thus $N(A^*) = \{0\}$ so A^* is one-to-one.

Case 1 : When $\det \Delta \neq 0$, what is $R(A)$?

Given $y \in X$. Suppose x is a solution of $Ax = y$.

Then

$$x - \sum_{k=1}^{n} x_k^*(x)x_k = y.$$

In order to solve

$$Ax = x - Kx = y,$$

it suffices to find Kx, that is, to find the scalars $x_1^*(x), x_2^*(x), \ldots, x_n^*(x)$. For each $j, 1 \leq j \leq n$,

$$x_j^*(x) - \sum_{k=1}^{n} x_k^*(x)x_j^*(x_k) = x_j^*(y).$$

Case 1 : When $\det\Delta \neq 0$, what is $R(A)$?

This implies that $\sum_{k=1}^{n}\Big\{\delta_{jk} - x_j^*(x_k)\Big\}x_k^*(x) = x_j^*(y), \quad 1 \leq j \leq n.$

Hence

$$\Delta \begin{pmatrix} x_1^*(x) \\ \vdots \\ x_n^*(x) \end{pmatrix} = \begin{pmatrix} x_1^*(y) \\ \vdots \\ x_n^*(y) \end{pmatrix}.$$

If $\det\Delta \neq 0$, the above system has a unique solution for $x_k^*(x), 1 \leq k \leq n$, and the solution x is unique because

$$x = y + \sum_{k=1}^{n} x_k^*(x)x_k.$$

Every $y \in X$ has a unique solution. Hence A is surjective.

Case 1 : When $\det \Delta = 0$, what is $N(A)$?

Suppose $x \in N(A)$. Then $x = Kx$, so $x = \sum_{j=1}^{n} \alpha_j x_j$, for some scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$.

$$\sum_{j=1}^{n} \alpha_j x_j = x = Kx = \sum_{j=1}^{n} x_j^*(x) x_j = \sum_{j=1}^{n} x_j^* \Big(\sum_{k=1}^{n} \alpha_k x_k \Big) x_j$$

which implies that $\displaystyle\sum_{j=1}^{n} \Big\{ \alpha_j - \sum_{k=1}^{n} \alpha_k x_j^*(x_k) \Big\} x_j = 0$. Since $\{x_1, x_2, \ldots, x_n\}$ is linearly independent, for each $j = 1, 2, \ldots, n$, we have $\alpha_j - \sum_{k=1}^{n} \alpha_k x_j^*(x_k) = 0$. Hence

$$\begin{pmatrix} 1 - x_1^*(x_1) & -x_1^*(x_2) & \cdots & -x_1^*(x_n) \\ -x_2^*(x_1) & 1 - x_2^*(x_2) & \cdots & -x_2^*(x_n) \\ \vdots & \vdots & \cdots & \vdots \\ -x_n^*(x_1) & -x_n^*(x_2) & \cdots & 1 - x_n^*(x_n) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Case 1 : When $\det \Delta = 0$, what is $N(A)$?

We have,

$$\Delta \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since $\det \Delta = 0$, we must have some non-zero solutions for α_j's. Note that $N(A) \subseteq Span\ \{x_1, x_2, \ldots, x_n\}$.

If the rank of Δ is $\ell < n$, then there are $n - \ell$ linearly independent solutions. Thus A is not one-to-one and the dimension of $N(A) = n - \ell$.

Case 1 : When $\det \Delta = 0$, what is $N(A^*)$?

Suppose $x^* \in N(A^*)$. Then $x^* = K^* x^*$, so $x^* = \sum_{j=1}^{n} \beta_j x_j$, for some scalars $\beta_1, \beta_2, \dots, \beta_n$.

$$\sum_{j=1}^{n} \beta_j x_j^* = x^* = K^* x^* = \sum_{j=1}^{n} x^*(x_j) x_j^* = \sum_{j=1}^{n} \Big(\sum_{k=1}^{n} \beta_k x_k^* \Big)(x_j) x_j^*$$

which implies that

$$\sum_{j=1}^{n} \Big\{ \beta_j - \sum_{k=1}^{n} \beta_k x_k^*(x_j) \Big\} x_j^* = 0.$$

Since $\{x_1^*, x_2^*, \dots, x_n^*\}$ is linearly independent, for each $j = 1, 2, \dots, n$,

$$\beta_j - \sum_{k=1}^{n} \beta_k x_k^*(x_j) = 0.$$

Case 1 : When $\det \Delta = 0$, what is $N(A^*)$?

Hence we have,

$$\Delta^T \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since $\det \Delta = \det \Delta^T = 0$, we must have some non-zero solutions for β_j's. Note that $N(A^*) \subseteq Span\ \{x_1^*, x_2^*, \dots, x_n^*\}$.

If the rank of Δ^T is $\ell < n$, then there are $n - \ell$ linearly independent solutions. Thus A^* is not one-to-one and the dimension of $N(A^*) = n - \ell$. Note that ranks of Δ and Δ^T are the same.

Case 1 : When $\det \Delta = 0$, what is $R(A)$?

Given $y \in X$. Suppose x is a solution of $Ax = y$.

Then

$$x - \sum_{k=1}^{n} x_k^*(x) x_k = y.$$

In order to solve

$$Ax = x - Kx = y,$$

it suffices to find Kx, that is, to find the scalars $x_1^*(x), x_2^*(x), \ldots, x_n^*(x)$. For each $j, 1 \leq j \leq n$,

$$x_j^*(x) - \sum_{k=1}^{n} x_k^*(x) x_j^*(x_k) = x_j^*(y).$$

Case 1 : When $\det \Delta = 0$, what is $R(A)$?

This implies that $\sum_{k=1}^{n} \left\{ \delta_{jk} - x_j^*(x_k) \right\} x_k^*(x) = x_j^*(y), \quad 1 \leq j \leq n.$

Hence

$$\Delta \begin{pmatrix} x_1^*(x) \\ \vdots \\ x_n^*(x) \end{pmatrix} = \begin{pmatrix} x_1^*(y) \\ \vdots \\ x_n^*(y) \end{pmatrix}. \tag{20}$$

If $\det \Delta = 0$, the above system (20) has many solution for $x_k^*(x), 1 \leq k \leq n$, and the solution x is not unique because

$$x = y + \sum_{k=1}^{n} x_k^*(x) x_k.$$

Case 1 : When $\det \Delta = 0$, what is $R(A)$?

If $\det \Delta = 0$, the above system (20) has many solution for $x_k^*(x), 1 \leq k \leq n$. **How to find these solutions?**

We recall a theorem (Linear Algebra, by A. Ramachandra Rao and P. Bhimasankaram, page 189) stated as follows:

Theorem 9.

The system $Ax = b$ is consistent iff

$$A^T u = 0 \implies b^T u = 0.$$

Case 1 : When $\det \Delta = 0$, what is $R(A)$?

In this case, (20) can be solved for those y which satisfy

$$\Delta^T \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = 0$$

implies

$$\begin{bmatrix} x_1^*(y) & x_2^*(y) & \cdots & x_n^*(y) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = 0.$$

That is, (20) can be solved for those y which satisfy

$$\sum_{j=1}^{n} \alpha_j x_j^*(y) = 0$$

whenever

$$\sum_{j=1}^{n} \Big[\delta_{jk} - x_j^*(x_k)\Big]\alpha_j = 0, \quad 1 \leq k \leq n.$$

Case 1 : When $\det \Delta = 0$, what is $R(A)$?

Now we claim that $R(A)$ is closed.

Operators "close" to operators of finite rank

We now think about operators which are "close" to operators of finite rank such that

$$\|K_n - K\| \to \text{ as } n \to \infty.$$

Theorem 10.

Let X be a Banach space, and assume that $K \in B(X)$ is the limit in norm of a sequence of operators of finite rank. If $A = I - K$, then $R(A)$ is closed in X, and $\dim N(A) = \dim N(A^*) < \infty$. FA-1(P-12)T-3

What kind of operators are the limits in norm of operators of finite rank?

If X is a Hilbert space, every compact operator is a limit in norm of operators of finite rank.

Also, every compact operator on many well-known Banach spaces, is a limit in norm of operators of finite rank.

If X is a Banach space, the hypotheses of the Theorem (10) may not be fulfilled for some $K \in K(X)$. However, we are going to show that, nevertheless, the conclusion is true.

Operators "close" to compact operators

Theorem 11.

Let X be a normed space and Y a Banach space. If L is in $B(X, Y)$ and there is a sequence $\{K_n\} \subseteq K(X, Y)$ such that

$$\|L - K_n\| \to \text{ as } n \to \infty$$

then L is in $K(X, Y)$.

Theorem 12 (Fredholm alternative).

Let X be a Banach space and let K be an operator in $K(X)$. Set $A = I - K$.

Then, $R(A)$ is closed in X and $\dim N(A) = \dim N(A^*)$ is finite.

In particular, either

$$R(A) = X \quad \text{and} \quad N(A) = \{0\}$$

or

$$R(A) \neq X \quad \text{and} \quad N(A) \neq \{0\}.$$

FA-1(P-14)T-4

To prove the theorem, we need the following results : Let X, Y be Banach spaces.

1. If $A \in B(X, Y)$ with $R(A) = Y, N(A) = \{0\}$. Then $A^{-1} \in B(X, Y)$.
2. If $\|A\| < 1$, then $I - A$ is invertible.
3. If $A \in B(X, Y)$, then $R(A)$ is closed if and only if there exists $C > 0$ such that
$$d(x, N(A)) \leq C\|Ax\|, \quad \text{for all } x \in X.$$
4. If A is a linear operator from X to Y, then for each x in X and $\varepsilon > 0$, there is an element x_0 in X such that
$$Ax_0 = Ax, \quad d(x_0, N(A)) = d(x, N(A))$$
and
$$d(x, N(A)) \leq \|x_0\| \leq d(x, N(A)) + \varepsilon.$$

Fredholm alternative

4. Let M be a proper closed subspace of a normed space X. Then for each number r satisfying $0 < r < 1$ there is an element $x_r \in X$ such that
$$\|x_r\| = 1 \quad \text{and } d(x, M) \geq r.$$
5. Let M be a subspace of a normed space X, and suppose that x_0 is an element of X satisfying $d = d(x_0, M) > 0$. Then there exists $x^* \in X^*$ such that
$$\|x^*\| = 1, \quad x_0^*(x) = d > 0$$
and
$$x^*(x) = 0, \quad \text{for all } x \in M.$$

Fredholm alternative

6. Let N be a subspace of X^*, and suppose that x_0^* is an element of X^* satisfying $d = d(x_0^*, N) > 0$. Then there exists $x \in X$ such that

$$\|x\| = 1, \quad x^*(x_0) = d > 0$$

and

$$x^*(x) = 0, \quad \text{for all } x^* \in N.$$

Fredholm Operators

If X is a Banach space and $K \in K(X)$, we have seen that $A = I - K$ has closed range and that both $N(A)$ and $N(A^*)$ are finite dimensional.

Operators having these properties form a very interesting class and arise very frequently in applications. They are called **Fredholm operators**.

Fredholm Operators

Definition 13.

Let X, Y be Banach spaces. An operator $A \in B(X, Y)$ is said to be **Fredholm operator** *from X to Y if*

1. $\alpha(A) =$ *dim* $N(A)$ *is finite,*
2. $R(A)$ *is closed in* Y,
3. $\beta(A) =$ *dim* $N(A^*)$ *is finite.*

The set of Fredholm operators from X to Y is denoted by $\Phi(X, Y)$.

The index of a Fredholm operator is defined as

$$i(A) = \alpha(A) - \beta(A).$$

If $X = Y$ and K is a compact operator on X, then $I - K$ is a Fredholm operator and $i(I - K) = 0$.

Semi-Fredholm Operators

For $A \in B(X, Y)$, if $R(A)$ is closed and $\alpha(A) < \infty$ (resp. $\beta(A) < \infty$), then A is called an **upper semi-Fredholm** (resp. **lower semi-Fredholm**) operator.

The set of all upper semi-Fredholm operators is denoted by $\Phi_+(X, Y)$ and the set of all lower semi-Fredholm operators is denoted by $\Phi_-(X, Y)$.

Upper or lower semi-Fredholm operators are called **semi-Fredholm operators.**

We shall discuss semi-Fredholm operators in the next lecture.

References

- Martin Schechter, "*Principles of Functional Analysis*," Second Edition, GSM 36, American Mathematical Society, Providence, Rhode Island, 2000. (Chapter 2 : pages mainly from 77 to 100).